Ex 6.2

6.2.1 $\quad I = \displaystyle\int_0^{\infty} \frac{\sin(x)}{x}\,dx \overset{\text{parité}}{=} 2\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\,dx$

$$I = \operatorname{Im}\left(\int_{-\infty}^{\infty} \frac{2}{x} e^{ix}\,dx\right), \text{ on pose } J = \int_{-\infty}^{\infty} \frac{2}{x} e^{ix}\,dx$$

On pose de plus : $H_R = \displaystyle\int_{\gamma_R} \frac{1}{z} e^{iz}\,dz$

$$H_R^1 = \int_{\pi}^{0} R e^{-i\theta} e^{i\frac{1}{R}e^{i\theta}} \frac{1}{R} i e^{i\theta}\,d\theta$$

$$H_R^2 = \int_{1/R}^{R} \frac{1}{x} e^{ix}\,dx \qquad H_R^3 = \int_0^{\pi} \frac{1}{R} e^{-i\theta} e^{iRe^{i\theta}} R i e^{i\theta}\,d\theta$$

$$H_R^4 = \int_{-R}^{-1/R} \frac{1}{x} e^{ix}\,dx$$

On a : $H_R^1 = -\displaystyle\int_0^{\pi} i e^{i\frac{1}{R}e^{i\theta}}\,d\theta$

Or : $\quad e^{i\frac{1}{R}e^{i\theta}} = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{i}{R}e^{i\theta}\right)^n$

D'où :

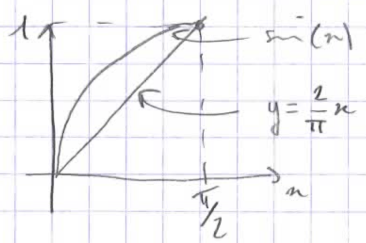
$$H_R^1 = -\int_0^{\pi} i \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{i}{R}e^{i\theta}\right)^n d\theta = -\sum_{n=1}^{\infty} \frac{(i)^{n+1}}{n!} \frac{1}{R^n} \frac{1}{in}\left[e^{i\theta n}\right]_0^{\pi} - i\pi$$

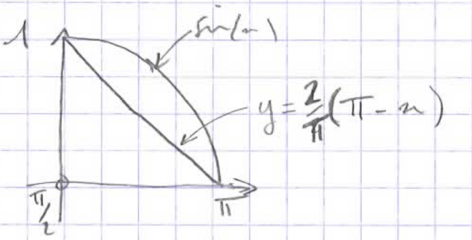
D'où on fait :

$$\lim_{R\to\infty} H_R^1 = -i\pi$$

De plus : $\quad |H_R^3| \leq \displaystyle\int_0^{\pi} e^{-\sin(\theta)R}\,d\theta$

On





D'où $\quad \sin(x) \geq \frac{2}{\pi}x \ \ \forall x \in [0, \frac{\pi}{2}] \quad , \quad \sin(x) \geq \frac{2}{\pi}(\pi - x) \ \ \forall x \in [\frac{\pi}{2}, \pi]$

Ainsi $\quad |H_R^3| \leq \displaystyle\int_0^{\pi/2} e^{-\theta\frac{2R}{\pi}}\,d\theta + \underbrace{\int_{\pi/2}^{\pi} e^{-\frac{2}{\pi}(\pi-\theta)R}\,d\theta}_{\tilde{\theta} = \pi - \theta} \Rightarrow \int_{\pi/2}^{0} e^{-\frac{2}{\pi}\theta R}(-d\theta)$

$$\leq -\frac{\pi}{2R}\left[e^{-\theta\frac{2R}{\pi}}\right]_0^{\pi/2} + \frac{\pi}{2R}\left[e^{-\theta\frac{2R}{\pi}}\right]_0^{\pi/2} \underset{R\to\infty}{\longrightarrow} 0$$

Finalement, on a, par le théorème des résidus :

$$H_R = \int_{\gamma_R} \frac{1}{z} e^{iz}\,dz = 0$$

$$\text{et} \quad \lim_{R\to\infty} \int_{\gamma_R} \frac{1}{z} e^{iz}\,dz = -i\pi + \int_{-\infty}^{\infty} \frac{1}{x} e^{ix}\,dx$$